Experimental Characterization of a

Josephson Parametric Amplifier

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Abstract

Recent years have seen significant advances in the development of quantum devices which employ superconducting circuitry. The fast pace of this progress has led to an increasing need for much higher accuracy in the measurement of quantum states in the microwave regime, with detection thresholds close to the quantum limit.

Work on the types of signal amplification devices that are able to operate in the low noise range approaching the quantum limit have recently been focusing on parametric amplifiers, which are able to achieve a high signal-to-noise ratio, albeit over a relatively narrow band of frequencies. By incorporating SQUID’s consisting of Josephson junctions, which provide the necessary nonlinearity for parametric amplification, the band of amplified frequencies, can be easily tuned over a wide range of microwave frequencies.

The current objective of the Superconducting Quantum Device Laboratory is to create a diverse toolbox of superconducting apparatuses with which the limits of quantum simulation and experimentation can be advanced. To add to this toolbox, the goal of this thesis is to characterize in detail one of the parametric devices that will be routinely used in qubit experiments and also to implement a practical compensator system, which will minimize the negative effects of the strong pump tone. This strong pump tone can saturate the detection system, increasing the signal to noise ratio, it can also cause the gain to deviate away from the target level, as well as propagating back and interfere with the signal source.

This thesis will start by reviewing the theoretical background of the Josephson parametric amplifier, including a literature survey of the relevant developmental timeline that led up to our current level of technology. I will then describe a simulated system that I have developed, which demonstrates an ideal microwave parametric amplifier circuit upon which the pump-cancellation procedure can be tested in simulation. I then adapt this simulated methodology to the laboratory environment and demonstrate the effectiveness of the cancellation routine. Finally I will give a detailed characterization of our team’s parametric amplifier, in terms of its frequency tuneability, Josephson junction characteristics, quality factors, driving power regimes, and its gain performance.
Introduction

The discovery of collective states of matter, in which macroscopic collections of atoms conform to an effective single-particle wave function, has opened up rich new avenues of exploration with regard to fundamental quantum principles. Over the last two decades, technological advances in our ability to control phenomena such as Bose-Einstein condensation and superconductivity has led to a period of intensive research into the use of these collective excitations to model coherent quantum dynamics.

One of the experimental methods that have allowed us the greatest degree of control has been the encoding of these quantum states into microwave fields, which can be incorporated into solid state or ‘on-chip’ apparatuses. These devices offer easy fabrication, reliable performance and a high degree of electric field control.

For these types of microwave circuits, the superconducting Josephson junction has been of pivotal significance. More specifically, the coherent dynamics exhibited by the two wave functions of the superconducting elements have been shown to constitute a macroscopic phase variable. The discrete electronic states of this collective variable have allowed the development of artificial atoms, to model real-world quantum states within a controllable superconducting circuit.

Nakamura et al. demonstrated that these artificial atoms could be strongly coupled to microwave photons in a superconducting circuit. This experiment used a Cooper box structure, in which a superconducting ‘island’ is coupled via a Josephson junction to a superconducting reservoir. This Cooper box constitutes a two-level charge qubit, in which the state is determined by the number of Cooper pairs tunnelling across the junction.

This area of research has led to rapid and intensive progress with topics such as superconducting qubits, nanomechanical resonators, quantum correlations, quantum teleportation, optical communication technology and Rydberg atoms. However, the innate weakness of the microwave photon and the dominance of thermal noise in electronic circuits required the development of detection and amplification technology which was not previously available in the high frequency microwave range, or at the milli-Kelvin operating temperatures of superconducting circuits.

In this ultra-low temperature regime quantum fluctuations are a much more dominant source of noise than thermal fluctuations, being in an energy scale where \( \hbar \omega > k_B T \). This means that when trying to design an amplifier that could make the weak signal of a quantum state measurement accessible to classical detection systems, quantum mechanical phenomena become our primary concern. It is this that has made the study of parametric amplifier devices once again an imperative for researchers in the field of quantum technology.
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1. The Mechanism of Josephson Parametric Amplification

A Nonlinear Resonator

Parametric amplifiers work by transferring photons from one frequency mode, at which a strong pump signal is supplying energy, into other frequencies at which the signal to be amplified is present. The Josephson parametric amplifier configuration consists of a microwave resonator, which has an oscillating boundary condition, as controlled by the nonlinear, time-varying impedance of the Josephson junction.

A Josephson junction is composed of two superconductors separated by an insulating barrier. Each superconducting wave function overlaps the other side, allowing a flow of electron-pairs to tunnel across the gap at a rate determined by their relative phases. This is the source of the nonlinear phase-current relation, \( I = I_c \sin \delta \), where \( I_c \) is the maximum or ‘critical’ current.

Since the phase difference is proportional to the junctions voltage: \( \frac{d\delta}{dt} = \frac{2e}{\hbar} V \), we can use the voltage relation, \( V = L \frac{dI}{dt} \), to determine the non-linear, time varying inductance:

\[
L_{ij} = \frac{\hbar}{2e I_c \sin \delta} \quad (1.1)
\]

This Josephson inductance regulates the oscillating boundary condition of the cavity. The nonlinearity of this boundary condition creates the necessary conditions for a resonator which conforms to the equation describing a driven Duffing oscillator:

\[
\ddot{x} + 2\gamma \dot{x} + \omega_o^2 x + \beta x^3 = F \cos \omega t \quad (1.2)
\]

As can be seen from this function, the dynamics of the system include a small, nonlinear cubic term, which is driven by the cosine term into increasingly strong regimes of nonlinear behaviour as the power of the pump tone is increased, this can be seen in figure 1.2. This indicates that the parametric oscillator operates in distinct power regions of linear (lorentzian shape, blue), non-linear (steep vertical response, orange) and bilinear response (multiple solutions, red). For strong gain in parametric amplifiers, we select a region which exhibits steep non-linearity.

The effect of this non-linear, oscillating boundary condition, is to add and subtract increments of phase to the eigenmodes inside the cavity. These increments create the necessary conditions for a process of inter-mode coupling between the pump mode and adjacent frequencies, this is the basis of four-wave mixing.

Fig1.1: Schematic of a Josephson Junction

Fig1.2: Linear, non-linear and bilinear response of a Duffing oscillator with increasing pump power.
The Four-Wave Mixing Process

Although the quantum devices we are characterising operate in the microwave regime, amplification through four-wave mixing is easiest to understand in quantum optical terminology. To amplify a weak optical signal, we would use a coherent pump source to drive a time-varying parameter in a non-linear medium. For instance a crystal medium with a $X^{(2)}$ or $X^{(3)}$ nonlinearity, exhibits a nonlinear polarization effect, which causes its refractive index to be dependent on the power of light that is passing through it. By driving this refractive index parameter using the pump tone, coupling can occur between the strong pump mode and other nearby frequencies, through a process known as either ‘three-wave’ or ‘four-wave’ mixing. Since the device we are characterising exhibits four-wave mixing, I will refer to this effect only.

As can be seen in figure 1.3, if two pump tones $2\omega_p$ and a signal $\omega_s$, are injected into a nonlinear medium, a two stage process results:

1. ‘Frequency difference’ mixing between $\omega_s$ and $\omega_p$ causes a new ‘idler’ tone at a frequency offset of $\omega_i = \omega_{p1} + (\omega_{p2} - \omega_s)$.

2. The same mixing process in reverse, results in an additional photon at the signal frequency $\omega_s = \omega_{p2} + (\omega_{p1} - \omega_i)$.

Since the original signal photon isn’t consumed by the mixing process, this results in amplification of the initial signal. Each time this four-wave mixing cycle occurs; two photons are transferred from the pump frequency and deposited in the signal and idler modes.

The four-wave mixing process follows essentially the same principle in the microwave regime, except with a nonlinear resonator as the medium and the electrical impedance of the Josephson junctions as the driven parameter.

When the pump tone is driving the resonator into a strong nonlinear regime, with inter-mode coupling and four-wave mixing taking place, we can consider a weak quantum signal as represented by an amplitude fluctuation added to the much stronger pump signal. In this regime even small perturbations will produce a disproportionately large phase change due to the nonlinear boundary condition, encoding the amplitude signal in an amplified phase response as can be seen in the amplifiers transfer function shown in figure 1.4.
When considered in quadrature space, the resulting difference in amplitude that occurs between the input signal vector and the output vector that has been shifted in phase, constitutes the degree of amplification that has been achieved. The dynamics of this amplitude and phase relation in quadrature space, is closely dependent on the mode of operation of the amplifier, this will be explained in greater detail in Section 2.

**Frequency Tunability of the SQUID Array**

While this parametric mechanism can exhibit gain well in excess of standard linear amplifiers, it achieves this over a relative narrow frequency bandwidth, often in the order of a few mega-Hertz. By utilising an array of Superconducting Quantum Interference Devices (figure 1.5) in place of single Josephson junctions, whose nonlinear inductance relation is sensitive to externally applied magnetic fields, we gain the ability to tune the resonant frequency of the resonator, to a bandwidth that is useful for practical experiments. This tunability factor can be seen in the SQUID inductance relation (equation 1.3), via the cosine flux term.

\[
L_s = \frac{\hbar}{4e I_c} \left| \cos \frac{\pi \phi}{\phi_0} \right| \cos \theta
\]

Where \( \phi \) is the external magnetic flux applied, \( \phi_0 \) is the magnetic flux quantum, \( \theta \) is the average phase between the two junctions \( \theta = (\delta_1 + \delta_2)/2 \) and \( I_c \) is the critical current above which the superconductivity of the Josephson junctions is suppressed and the SQUID stops operating.

The use of multiple SQUIDs in an array, also has the added benefit of increasing the total impedance and power of the lumped SQUID element, allowing pump tones of higher power to be used without exceeding the junctions critical current. This is a necessary requirement to allow the system to be driven with high power into areas of higher signal gain.
2. Parametric Amplifier Modes of Operation

The Standard Quantum Limit

The concept of a lowest detection limit beyond which we cannot measure due to quantum mechanical noise mechanisms was described by Haus and Mullen\textsuperscript{4} and further derived by Caves\textsuperscript{5}, primarily based upon the Heisenberg uncertainty principle, which limits the simultaneous measurement of conjugate variables. These authors concluded that a ‘Standard Quantum Limit’ (SQL) exists for all amplification that includes the simultaneous measurement of non-commuting observables, which in our case refers to the in-phase and in-quadrature components of a measured signal.

Following the Caves explanation, if we had a signal that we wanted to amplify, it could be expressed in terms of quadrature amplitude components:

\[ E(t) \propto (a e^{-i\omega t} + a^* e^{i\omega t}) \]  

Equation (2.1)

In the case of quantum signal amplification, these amplitudes would constitute creation and annihilation operators \( a \rightarrow \hat{a} \) and \( a^* \rightarrow \hat{a}^\dagger \). These observable operators form a conjugate pair which satisfy the commutation relation: \( [\hat{a}, \hat{a}^\dagger] = 1 \).

From a practical perspective, when we are attempting to amplify a quantum signal, we are primarily interested in how the output signal relate to the input signal modes. This can be defined using output operators, \( \hat{b} \) and \( \hat{b}^\dagger \) which must also obey the commutation relation above. Assuming that for the moment we are going to be amplifying each quadrature equally, we can define a gain coefficient \( G \), giving the relations:

\[ \hat{b} = \sqrt{G} \hat{a} \quad , \quad \hat{b}^\dagger = \sqrt{G} \hat{a}^\dagger \]  

Equation (2.2)

These amplified operators do not satisfy the commutation relation, \( [\hat{b}, \hat{b}^\dagger] \neq 1 \), but could be corrected by the inclusion of an additional operator of some form to equalise the commutator imbalance.

\[ \hat{b} = \sqrt{G} \hat{a} + \hat{F} \quad , \quad \hat{b}^\dagger = \sqrt{G} \hat{a}^\dagger + \hat{F}^\dagger \]  

Equation (2.3)

It is the necessity for this additional operator that is the source of the extra noise which gives rise to the quantum limit on amplifier performance.

Parametric Amplifier Modes of Operation

The parametric amplifier can be operated in two regimes, known as ‘phase preserving’ or ‘phase sensitive’ modes. I will be following a visual quadrature space explanation similar to that used by Castellanos-Beltran\textsuperscript{25}, which demonstrates how small quantum signals that act like a perturbation on top of the much larger pump signal can cause disproportionate fluctuations of the reflected phase of the pump signal. This encoding of the weak signal into the phase change of the strong signal is directly related to amplification as, when there is a large oscillation in pump phase (eg. Fig2.2a) the vector difference between the output states and the input states, is large, which directly represents a power gain. This power gain occurs at the frequency of the modulating input signal, but with a proportion of the amplitude of the much stronger pump signal, which constitutes amplification.
1. ‘Phase preserving’ mode:
Refers to the instance in which both quadratures are amplified equally. The parametric amplifier can produce substantial gain over a narrow bandwidth, and operate very close to the standard quantum limit, but will still be constrained by the uncertainty principle. In quadrature space this mode appears as a small signal vector which rotates in phase as the in-phase and in-quadrature components vary, this rotation in turn causes the large pump vector to oscillate in phase at the same frequency, as displayed in figure 2.1.

Fig.2.1: Phase-preserving mode; the signals phase rotation causes oscillations in the pump phase.

2. ‘Phase sensitive’ mode:
In this mode the in-phase and in-quadrature components are de/amplified separately by different factors. This allows us to create a range of output operators that could satisfy the commutation relations without the need for the additional noise-operator that was described above. An example of a possible set of output annihilation/creation operators is:

\[
\hat{b} = \sqrt{G} \hat{a}, \quad \hat{b}^\dagger = \frac{1}{\sqrt{G}} \hat{a}^\dagger
\]

This would describe an instance where one quadrature of the signal was de-amplified by the same proportion that the other quadrature was amplified, theoretically without any quantum noise being present in the amplified component. This effect can be seen in figure 2.2, in which each of the quadrature components interact separately with the pump signal and have a constant phase angle, rather than the rotating phase frequency exhibited earlier.

Fig.2.2: Phase-sensitive mode; a) In-phase signal adds coherently, causing a disproportionate phase variation in the pump output b) In-quadrature signal causes negligible phase-related difference in pump output vectors.
When the amplifier is pumped in a regime with a steep power-phase transfer relation, the small coherent addition of the input signal can cause a much larger phase oscillation in the output pump signal, as can be seen in Figure 2.2a. As explained previously, the amplitude of power that has been shifted to the signal frequency is demonstrated by the vector difference between the pump vectors at input and the output vectors. For the In-phase component this difference is very large indicating significant amplification, however for the In-quadrature component, it is small to the point of negligibility.

The characterization that I will be performing in this thesis will concentrate on the phase-preserving mode of operation, providing practical system information and compensator solutions that our research team can use to run the parametric amplifier effectively. However it is the potential for noiseless amplification, using phase-sensitive methods that have the possibility of surpassing the SQL, which has driven a resurgence of interest in parametric amplifiers, particularly with regard to applications in the context of superconducting devices.
3. Input-Output Theory

A Coupled, Linear Transmission Line Resonator

The parametric amplifier that we are using is at its most simple, a nonlinear version of the standard transmission line resonator that is often used in quantum device microwave circuits. In our experiment we will be using the coplanar waveguide type of resonator, which consists of a straight length of superconducting niobium of carefully fabricated dimensions, which have semi-infinite ground planes running parallel, fabricated onto a dielectric substrate, such as sapphire (see figure 3.1).

The mathematical approach that is most pertinent to my characterization of the parametric amplifier is to use an input-output method to analyse the reflected output signal of the amplifier as it is probed by an input microwave signal. Figure 3.2 shows a conceptual schematic of the system, combining an environmental bath of infinite loss modes, as represented by an effectively infinite transmission line, which is coupled via a coupling capacitor to the transmission line resonator. For simplicity, I will refer to only one of the modes in the resonator system, a_n.

The Hamiltonian for the parametric amplifier is: $H_{\text{system}} + H_{\text{bath}} + H_{\text{coupling}}$, which combines:

- The transmission line resonator, as modelled by the standard Hamiltonian for a series of independent LC oscillators, but incorporating an infinite series of possible frequency modes.

$$H_{\text{system}} = \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{q_n^2}{C_n} + \frac{\dot{\varphi}_n^2}{L_n} \right)$$  \hspace{1cm} (3.1)

Where $\varphi_n$ is the magnetic flux variable of the normal modes inside the resonator and $q$ is its conjugate the charge variable, $q_n = C_n \dot{\varphi}_n$, and $C_n$ & $L_n$ are respectively the resonators capacitance and inductance.

- The Hamiltonian which describes the bath of environmental modes to which the resonator is coupled. This includes loss, mode decoherence caused by environmental coupling, to electromagnetic and phonon modes within the sample material. This bath is modelled as an effectively infinite length of transmission line with the Hamiltonian:

$$H_{\text{bath}} = \hbar \int_{\omega} d\omega \omega b_\omega^\dagger b_\omega + 1/2$$  \hspace{1cm} (3.2)
Where the mode frequency $\omega$ is a continuous variable, $b / b^\dagger$ indicate the input modes from outside the resonator and the constant factor of $\frac{1}{2}$ represents the vacuum contribution which can be neglected, as it doesn’t affect the dynamics that we are interested in.

- The coupling capacitance between the bath and the system, which has the Hamiltonian:

$$\mathcal{H}_{\text{coupling}} = \hbar \sqrt{\kappa/2\pi} \int_{\omega} d\omega \left( b^\dagger_a \omega a + b_\omega a^\dagger \right)$$

(3.3)

Which includes the interaction of the input modes, $b$ and the resonator modes, $a$. The $\kappa$ factor is the coupling coefficient of the capacitor.

The equations of motion for this combined system Hamiltonian, are of the form:

$$\dot{a}(t) = -\frac{i}{\hbar} [a, \mathcal{H}_{\text{sys}}] - \frac{\kappa}{2} a(t) + \sqrt{\kappa} b_{\text{in}}(t)$$

(3.4)

With the boundary condition between input modes and coupled output modes given by:

$$b_{\text{out}}(t) = \sqrt{\kappa} a(t) - b_{\text{in}}(t)$$

(3.5)

This equation of motion can be understood in terms of its three components, which are respectively: a) the unitary evolution of the resonator Hamiltonian b) loss term due to the coupling of the resonator modes to the bath and c) a pumping term, indicating that input signals are injecting energy into the system.

**The Driven, Non-Linear Oscillator Model**

The equation of motion in equation 3.4, refers to the linear behaviour of a transmission line resonator, but to describe the nonlinear behaviour of the parametric amplifier, we must include a Kerr nonlinearity $K$, which represents the vital nonlinear behaviour of the amplifiers Josephson junctions. The revised Hamiltonian is then given by:

$$\mathcal{H}_{\text{JPA}} = \hbar \tilde{\omega}_0 A^\dagger A + \hbar \frac{K}{2} (A^\dagger)^2 A^2$$

(3.6)

Substituting this Hamiltonian into the equation of motion for the linear resonator example we can derive the non-linear resonators equation of motion:

$$\dot{A} = -i \omega_o A - i K A^\dagger A A - \frac{(\kappa + \gamma)}{2} A + \sqrt{\kappa} A_{\text{in}}(t) + \sqrt{\gamma} b_{\text{in}}(t)$$

(3.7)

Where $\gamma$ is a coupling coefficient indicating the internal losses of the resonator in a non-ideal system.

As in the linear resonator example we can identify boundary conditions for both the coupling coefficient and loss coefficient of the form:

$$A_{\text{out}}(t) = \sqrt{\kappa} A(t) - A_{\text{in}}(t)$$

(3.8)
As previously explained, the parametric amplifier system is both driven by a strong classical pump tone and probed by a weak quantum signal, to be amplified. Therefore if we break down the input modes into their pump and signal components, we can separate the inputs’ quantum and classical components.

\[ A_{\text{in}}(t) = (a_{\text{in}}(t) + \alpha_{\text{in}})e^{-i\omega_{p}t} \]
\[ A_{\text{out}}(t) = (a_{\text{out}}(t) + \alpha_{\text{out}})e^{-i\omega_{p}t} \]
\[ A(t) = (a(t) + \alpha)e^{-i\omega_{p}t} \]

(3.9)

Where the \( \alpha \) terms are the classical pump field components and the \( a \) terms indicate the quantum signal fields. We also use a frame which takes advantage of the rotating frame at the frequency of the driving pump, as shown by the exponential term.

Substituting these components into equation 3.7, and retaining only the classical \( \alpha \)-factors that refer to the pump field, we can create a steady-state solution by setting \( \dot{\alpha} = 0 \).

\[ \left( i(\tilde{\omega}_{0} - \omega_{p}) + \frac{\kappa + \gamma}{2} \right) \alpha + iK\alpha^{2}\alpha^{*} = \sqrt{\kappa} \alpha_{\text{in}} \]

(3.10)

Multiplying this equation with its complex conjugate allows us to create pairs of \( \alpha \) operators, that we can relate directly to the mean number of photons in the resonator.

\[ \left( \frac{\omega_{p} - \tilde{\omega}_{0}}{\kappa + \gamma} \right)^{2} + \frac{1}{4} |\alpha|^{2} - 2\left( \frac{\omega_{p} - \tilde{\omega}_{0}}{\kappa + \gamma} \right) K |\alpha|^{4} + \left( \frac{K}{\kappa + \gamma} \right) |\alpha|^{6} = \frac{\kappa}{(\kappa + \gamma)^{2}} |\alpha_{\text{in}}|^{2} \]
\[ \left( \frac{\omega_{p} - \tilde{\omega}_{0}}{\kappa + \gamma} \right)^{2} n - 2\left( \frac{\omega_{p} - \tilde{\omega}_{0}}{\kappa + \gamma} \right) K n^{2} + \left( \frac{K}{\kappa + \gamma} \right) n^{3} = \frac{\kappa}{(\kappa + \gamma)^{2}} n \]

(3.11)

To simplify this equation, we define scale invariant factors that we will use as the main operating parameters when characterizing the parametric amplifier:

- The detuning between the pump and resonator frequency in units of the resonators linewidth \((\kappa + \gamma)\).
  \[ \delta \equiv \frac{\omega_{p} - \tilde{\omega}_{0}}{\kappa + \gamma} \]

- The drive amplitude, expressed in a unitless form.
  \[ \bar{\alpha}_{\text{in}} \equiv \frac{\sqrt{\kappa} \alpha_{\text{in}}}{(\kappa + \gamma)} \]

- The product of drive power and the Kerr nonlinearity factor \( K \).
  \[ \xi \equiv \frac{|\bar{\alpha}_{\text{in}}|^{2}K}{(\kappa + \gamma)} \]

- The mean number of photons in the resonator, relative to the input driving power.
  \[ n \equiv \frac{|\alpha|^{2}}{|\alpha_{\text{in}}|^{2}} \]
Substituting these factors into equation 3.11 gives us the simplified equation.

\[ 1 = \left( \delta^2 + \frac{1}{4} \right) n - 2\delta \xi n^2 + \xi^2 n^3 \]  

(3.12)

This cubic equation allows us to derive a purely analytical solution for \( n \) in terms of \( \delta \) and \( \xi \), using a Mathematica solver routine. This solution is significant because it relates the average number of photons in the system, to both the frequency dependent character of the resonator and also the nonlinearity and power behaviour of the Josephson junction boundary. It plays an integral role to the dynamics of the system simulation that I will be describing in subsequent chapters.

**The Reflection Coefficient**

When using and characterizing the parametric amplifier, we are concerned with the input-output nature of the system, which is primarily described by its \( S_{11} \) scattering parameter, the complex reflection coefficient. Concentrating on the classical pump field for the moment, we can substitute the input-output relation, \( \alpha_{\text{out}} = \sqrt{\kappa} \alpha - \alpha_{\text{in}} \), into equation 3.10 and rearrange to find the relation:

\[ \Gamma = \frac{\alpha_{\text{out}}}{\alpha_{\text{in}}} = \frac{\kappa}{(\kappa + \gamma) \left( \frac{1}{2} - i \delta + i \xi n \right)} - 1 \]  

(3.13)

**Calculating the Kerr Constant of Nonlinearity**

We can experimentally measure or derive many of the scale invariant factors that make up the reflection coefficient equation (3.13). However, one of the important factors that will be required to fully characterise the parametric amplifier involves calculating the Kerr nonlinearity constant \( K \), which defines much of the systems power and gain dynamics. We can use the values derived above for the Josephson energy and inductance, to determine the \( K \) constant, which will also allow us to compare our experimental results to the theoretical model.

Following a method briefly outlined in Eichler\textsuperscript{23}, the \( K \) constant, can be determined for a specific mode within the resonator from the relation:

\[ K_i \approx -\frac{E_i}{2 \hbar} \left( \frac{\phi_i}{\varphi_0} \right)^4 \left( \frac{l_i}{d} \right)^4 \cos^4(k_i d) \]  

(3.14)

Where the \( k_i \) term indicates the resonators normal modes, \( \phi_i \) is the magnetic flux variable introduced in equation 3.1, given by \( \phi = \sqrt{\hbar/2\omega_0 C_0} \), and \( C_0 \) is the capacitance of the resonator in the fundamental mode. This transmission line capacitance can be calculated in the linear limit using the normal mode equation detailed by Wallquist et al.\textsuperscript{46}.

\[ C_i = c \int_0^d dx \cos^2(k_i x) = \frac{cd}{2} \left( 1 + \frac{\sin(2k_i d)}{2k_i d} \right) \]  

(3.15)
Where \( c \), the capacitance per unit length of the transmission line resonator, can be calculated from our linear measurements of the resonators frequency, and the transmission lines impedance (50\( \Omega \)), in the limit that the SQUID inductance goes to zero:

\[
C_i \xrightarrow{L_j \to 0} \frac{cd}{2} = \frac{\pi}{4Z_0\omega_{\text{reson}}} \tag{3.16}
\]

As we are primarily concerned with the central fundamental mode, in which the parametric amplification effect occurs, we can use a first order approximation of equation 7.2 given by:

\[
k_0 \approx \frac{k_0^{(0)}}{1 + L_j/ld} \tag{3.17}
\]

In which the \( k_0^{(0)} \) term indicates the \( L_j \to 0 \) limit of this zero mode: \( k_0^{(0)} d = \frac{\pi}{2} (1 + 2j) \), for \( j = \{0,1,2,... n\} \).

This allows us to simplify the coefficient expression into the form of equation 7.7:

\[
K_0 \approx \left[ -\frac{E_j}{2\hbar} \left(\frac{\phi}{\varphi_0}\right)^4 \left(\frac{L_j}{ld}\right)^4 \right] / \sqrt{N_{\text{SQUID}}} \tag{3.18}
\]

Where, \( N_{\text{SQUID}} \) is the number of SQUID loops in the amplifiers array.

**The Quantum Signal Gain Equations**

Now that we have found a relation which describes the classical pump field response of the parametric amplifier, we can describe what happens to the weak quantum signal field and derive the equations which define the gain behaviour of the amplifier.

Once again we substitute the input relations (equations 3.9) into the equation of motion for the intra-resonator field (equation 3.7), but this time we select only those factors which include the quantum signal field, \( \alpha \). In addition, because the quantum signal is so much smaller than the pump field, we linearise the resulting equation, disregarding any term which includes factors such as \( \alpha^* \alpha \), and thus keep only terms that are linear in \( \alpha \), leaving the relation:

\[
\dot{\alpha}(t) = i \left( \omega_p - \bar{\omega}_a - 2K|\alpha|^2 + i \left(\frac{\kappa + \gamma}{2}\right) \right) \alpha(t) - iK\alpha^2 \alpha^*(t) + \sqrt{\kappa} a_{\text{in}}(t) + \sqrt{\gamma} b_{\text{in}}(t) \tag{3.19}
\]

Where by convention we would need to use \( \alpha = |\alpha|e^{i\phi} \) to incorporate an initial phase to the driving pump signal, as a kind of phase reference point for the amplifier system. By using the Fourier decomposition method to break the different modes into their harmonic components, we can substitute the components:

\[
\alpha(t) \equiv \sqrt{\frac{(\kappa + \gamma)}{2\pi}} \int_{-\infty}^{\infty} d\Delta e^{-i(\kappa + \gamma)t} \delta_\Delta \tag{3.20}
\]
\[ a_{\text{in}, \Delta}(t) \equiv \sqrt{\frac{(\kappa + \gamma)}{2\pi}} \int_{-\infty}^{\infty} \Delta e^{-i\Delta(\kappa + \gamma)t} a_{\text{in}, \Delta} \]

\[ b_{\text{in}, \Delta}(t) \equiv \sqrt{\frac{(\kappa + \gamma)}{2\pi}} \int_{-\infty}^{\infty} \Delta e^{-i\Delta(\kappa + \gamma)t} b_{\text{in}, \Delta} \]

Substituting these Fourier components into the linearised equation of motion, we can select out a relation consisting of the coefficients of each of the harmonics which could exist.

\[ 0 = \left( i(\delta - 2\xi n + \Delta) - \frac{1}{2} \right) a_{\Delta} - i\xi n e^{2i\phi} a_{-\Delta} + \frac{\sqrt{\kappa}}{(\kappa + \gamma)} + \frac{\sqrt{b_{\text{in}, \Delta}}}{(\kappa + \gamma)} (3.21) \]

Where \( \Delta \) is the signal detuning from the pump frequency, in units of linewidth. As can be immediately seen, the equation combines signal modes from the detuning at frequency + \( \Delta \) and also at - \( \Delta \). This is the mathematical description of a physical wave-mixing process that occurs within the nonlinear resonator.

In addition the last two terms of the expression indicate the total modes that are injected into the parametric amplifier from outside the system, allowing us to separate them out and derive an expression for how these detuned modes react to incoming signal fields. First we define the incoming mode term:

\[ \tilde{c}_{\text{in}, \Delta} \equiv \frac{\sqrt{\kappa}}{(\kappa + \gamma)} + \frac{\sqrt{b_{\text{in}, \Delta}}}{(\kappa + \gamma)} (3.22) \]

By rearranging and substituting equation (3.21) into this definition, we can construct a matrix which relates the incoming modes at the detunings \( \pm \Delta \) to the resonator modes at those same detunings.

\[ \begin{bmatrix} \tilde{c}_{\text{in}, \Delta} \\ \tilde{c}_{\text{in}, -\Delta}^\dagger \end{bmatrix} = \begin{bmatrix} i(-\delta + 2\xi n - \Delta) + \frac{1}{2} & i\xi n e^{2i\phi} \\ -i\xi n e^{2i\phi} & i(\delta - 2\xi n - \Delta) + \frac{1}{2} \end{bmatrix} \begin{bmatrix} a_{\Delta} \\ a_{-\Delta}^\dagger \end{bmatrix} (3.23) \]

Since we are only really interested in the amplification process that occurs at the experimental detuning \( \Delta \), we can invert the coefficient matrix and identify an equation that just relates the incoming signals to the excitations of the intra-resonator signal modes.

\[ a_{\Delta} = \frac{i(\delta - 2\xi n - \Delta) + \frac{1}{2}}{(i\Delta - \lambda_{-})(i\Delta - \lambda_{+})} \tilde{c}_{\text{in}, \Delta} + \frac{-i\xi n e^{2i\phi}}{(i\Delta - \lambda_{-})(i\Delta - \lambda_{+})} \tilde{c}_{\text{in}, -\Delta}^\dagger (3.24) \]

Where \( \lambda_{\pm} = \frac{1}{2} \pm \sqrt{(\xi)^2 - (\delta - 2\xi n)^2} \)

By starting with the boundary condition equation 3.8, substituting this new equation for \( a_{\Delta} \) (3.24) and resubstituting definition 3.22, to remove the incoming mode factors \( \tilde{c}_{\text{in}, \Delta} \), we can finally create an equation relating the input and output modes of the resonator at detunings \( \pm \Delta \).
\[
\begin{align*}
a_{\text{out},\Delta} &= 
\left[\frac{\kappa}{(\kappa + \gamma)} - \frac{i(\delta - 2\xi n - \Delta) + \frac{1}{2}}{(i\Delta - \lambda_-)(i\Delta - \lambda_+)} - 1\right] a_{\text{in},\Delta} + 
\left[\frac{\kappa}{(\kappa + \gamma)(i\Delta - \lambda_-)(i\Delta - \lambda_+)} - \frac{-i\xi n e^{2i\phi}}{2}\right] a^\dagger_{\text{in},-\Delta} \quad (3.25) \\
&\phantom{=} + \sqrt{\frac{\kappa}{(\kappa + \gamma)(i\Delta - \lambda_-)(i\Delta - \lambda_+)} - \frac{i\xi n e^{2i\phi}}{2}} b_{\text{in},\Delta} + 
\left[\frac{\kappa}{(\kappa + \gamma)(i\Delta - \lambda_-)(i\Delta - \lambda_+)} - \frac{-i\xi n e^{2i\phi}}{2}\right] b^\dagger_{\text{in},-\Delta}
\end{align*}
\]

Which in the ideal no-loss case becomes:

\[
\begin{align*}
a_{\text{out},\Delta} &= 
\left[\frac{\kappa}{(\kappa + \gamma)} - \frac{i(\delta - 2\xi n - \Delta) + \frac{1}{2}}{(i\Delta - \lambda_-)(i\Delta - \lambda_+)} - 1\right] a_{\text{in},\Delta} + 
\left[\frac{\kappa}{(\kappa + \gamma)(i\Delta - \lambda_-)(i\Delta - \lambda_+)} - \frac{-i\xi n e^{2i\phi}}{2}\right] a^\dagger_{\text{in},-\Delta} \quad (3.26) \\
&\phantom{=} + \sqrt{\frac{\kappa}{(\kappa + \gamma)(i\Delta - \lambda_-)(i\Delta - \lambda_+)} - \frac{i\xi n e^{2i\phi}}{2}} b_{\text{in},\Delta} + 
\left[\frac{\kappa}{(\kappa + \gamma)(i\Delta - \lambda_-)(i\Delta - \lambda_+)} - \frac{-i\xi n e^{2i\phi}}{2}\right] b^\dagger_{\text{in},-\Delta}
\end{align*}
\]

When we consider the physical description of the wave mixing process that occurs between the detunings \(\pm \Delta\), it is clear that we can identify the coefficients as ‘signal’ and ‘idler’ gain terms.

\[
g_{S,\Delta} = -1 + \frac{\kappa}{(\kappa + \gamma)(i\Delta - \lambda_-)(i\Delta - \lambda_+)} \cdot \frac{i(\delta - 2\xi n - \Delta) + \frac{1}{2}}{\kappa} \quad (3.27)
\]

So in the lossless example, the output of the amplifier is a mixture of frequency modes at detunings \(\pm \Delta\), amplified by the respective gain coefficients, \(g_{S,\Delta}\) and \(g_{I,-\Delta}\).

\[
a_{\text{out},\Delta} \gamma \rightarrow 0 = g_{S,\Delta} a_{\text{in},\Delta} + g_{I,-\Delta} a^\dagger_{\text{in},-\Delta} \quad (3.28)
\]

Both mathematically in the equations above and theoretically in the early description of the three wave mixing process, it can be observed that these signal and idler gain terms conform to the total gain relation:

\[
G_{\Delta} \equiv \left|g_{S,\Delta}\right|^2 = \left|g_{I,-\Delta}\right|^2 + 1 \quad (3.29)
\]

Which demonstrates the nature of the wave-mixing process, where signal photons are mixed with idler frequencies without the initial signal photons being ‘consumed’, causing the signal gain to always be higher than the idler gain.
4. The Development of Josephson Parametric Amplifiers

Early Experimental Implementations

Superconducting Parametric Amplifiers based on nonlinear Josephson inductance were first demonstrated in 1975, by Feldman et al. The design consisted of a tin waveguide that ended in a series of 80 unbiased Josephson Junctions, which were then coupled to a $\lambda/4$ open-ended resonator, see Figure 4.1.

In this way input signals from the circulator interacted with the non-linear resonator array and were reflected out via the circulator, once the tuning stub had removed higher order harmonics. By operating at a temperature of 2.5K, well below the critical temperature of tin, they observed gain at a point around 10GHz.

This promising experiment was followed by similar designs throughout the 1980’s and 1990’s, refining key aspects of the apparatus. Silver et al demonstrated that by replacing the individual Josephson Junctions with SQUID junctions, desirable operating features could be achieved such as phase bias stability, reduced low frequency noise due to phase rectification and a significantly increased range over which the pump signal will drive the phase. In addition he made the observation that by including arrays of SQUIDs in series, rather than a single junction, the array could be designed to operate with exactly the same features as a single junction but would have greatly increased overall impedance meaning the total energy of the array could be increased. This is an important factor as it allows the parametric amplifier to be driven to much higher powers, making new regions of nonlinearity accessible, with a corresponding potential to increase the gain of the device.

However, the experiments of this time were marked by a phenomenon known as “noise rise”, in which the amplifiers noise temperatures were significantly greater than theoretically predicted, and exhibited sharp resonant peaks of noise, throughout the bias range. At the time, this phenomenon was attributed to factors such as the sensitivity of the parametric amplifiers operating point to fluctuations, or chaotic behaviour due to nonlinear dynamics.

For this reason, along with a low dynamic range and narrow bandwidth, in the order of 1MHz, parametric amplifiers were not widely used in technological applications. However, research continued and in 1985 an experiment by Smith, Sandell, Burch and Silver, observed a noise temperature of 6K that was sufficiently low to allow the detection of squeezed states of light, a necessary first step towards phase-sensitive, quantum-limited performance.

Subsequent research by Yurke et al. demonstrated the squeezing effect, achieving a noise temperature of 0.28K, but a gain of only 12dB before noise-rise phenomena interfered with the amplification. Their ongoing work over the next few years clarified the role of noise rise in the apparatus, demonstrating the theoretical work of Bryant, Wiesenfeld and McNamara, that the system reached a Hopf bifurcation point as the pump power rose within the nonlinear resonator. The Yurke et al. team reached new noise lows, observing squeezing effects that were 47% below the vacuum noise floor, near to the quantum limit and demonstrating substantial amplifier gains of up to 37dB at a noise temperature of 0.446K.
Recent Advances in JPA Design

Despite these advances, it wasn’t until the last decade that parametric amplifiers once again became an active area of intensive research. This interest is primarily due to the rapid advances in applications such as superconducting qubit readouts\textsuperscript{16,17}, measurements of nano-electromechanical motion\textsuperscript{18,19}, and optical communication technology\textsuperscript{20,21}.

A profusion of Josephson parametric amplifiers have been developed in the last few years but the transmission line resonator design which I will be characterising in this paper is one that has received considerable scrutiny for a range of quantum information and signal measurement applications.

As can be seen from the circuit schematic in Fig.4.2, the device as characterised by Castellanos-Beltran et al\textsuperscript{22} was similar to the original Feldman design, capacitively coupling a transmission line resonator, which incorporate an array of SQUID junctions that are end with a short circuit. The signal and pump tones are combined before being reflected off the amplifiers coupling capacitor, where the signals interact with the nonlinear oscillating mode, indicated in the schematic by a combined LRC circuit (blue dashed box). This design proved to be tuneable over a range of 4-7.8GHz and indicated gains of 28dB, outperforming linear HEMT amplification by 16dB.

The Castellanos-Beltran et al. group also worked on a design which injected the pump tone via the previously short-circuit end of the SQUID array\textsuperscript{23}, as can be seen in Fig.4.3. This design has the potential to take measurements from either end of the device, allowing direct measurement of both the reflected $S_{11}$ and transmitted $S_{21}$ scattering parameters, increasing the amount of data from which information could be inferred and allowing finer design control of the pump-resonator and signal-resonator coupling coefficients.

This design provided amplification gain of around 37dB, with a bandwidth in the order of 2 MHz, tuneable over the range 4-8GHz, and was used to demonstrate signal squeezing phenomena\textsuperscript{24}. In the same year, similar “two port” apparatus’ were demonstrated by Kinion et al\textsuperscript{26} and Palacios-Laloy et al\textsuperscript{27}.

Later work by Eichler and Wallraff\textsuperscript{28,29}, used the single-ended, reflected-signal design, incorporating a secondary pump line, which once appropriately calibrated, is able to cancel out the strong pump tone, which can cause interference at the signal source and saturation of the measurement sensors. The apparatus reliably achieved greater than 30dB of gain which could be detuned to broaden the bandwidth, when needed.
Eichler et al. demonstrated the design's capacity for generating two-mode squeezed states and measured a reduction of quantum noise of around -2dB below the standard quantum limit\textsuperscript{30}. Due to the broad range of accessible resonance frequencies and the reliable tunability of its gain and bandwidth characteristics, it is already being used in quantum information applications. Recent work includes its use as a qubit readout\textsuperscript{31}, to amplify qubit signals in quantum teleportation\textsuperscript{32,33}, and to prepare squeezed states for use in entanglement\textsuperscript{34} and quantum state tomography experiments\textsuperscript{35}.

**Variations On The Principle**

The recent interest in the field of Josephson parametric amplifiers has given rise to a diverse alternatives to the standard transmission line resonator design.

A recent design called the “Josephson ring modulator” spatially separates the signal and idler tones in the output channel, which has proved highly useful in the field of quantum information, providing correlated photons for entanglement experiments\textsuperscript{36-38}.

Yamamoto et al in 2008\textsuperscript{39}, produced a “Flux driven” Josephson parametric amplifier, which is a variant on the standard transmission line resonator design, which introduces the pump tone into the resonator system by driving the magnetic flux which is applied to the SQUID array. This design has the valuable benefit of isolating the amplified signal from the saturating effects of the strong pump tone, which removes the need for tone cancellation in the output circuitry.

As described in the introduction, the Josephson parametric amplifier can be driven into different operating regimes, by increasing the pump power. In 2008, Siddiqi et al\textsuperscript{40-41} developed a “Josephson bifurcation amplifier” which exploits the strong discontinuity that occurs, beyond the systems Hopf bifurcation point, in the bilinear region.

The magnetic sensitivity of the SQUIDs used in parametric amplifiers, has been used by Hatridge et al\textsuperscript{42}, to produce extremely high sensitivity readouts for magnetometry. By passing a continuous microwave signal through a parametric amplifier and measuring the amplified fluctuations that occur due to external flux, smaller magnetic fields can be detected than the standard, high sensitivity SQUID magnetometers.

Research has also focussed on alternative Josephson junction designs including a “weak link” junction, which constitutes a bridged narrowing of the waveguide, as opposed to the tunnelling junction. These junctions have been demonstrated to conform to the existing theoretical model based on a Kerr nonlinearity\textsuperscript{43}, exhibit reduced insulator-dissipation noise, and may improve coherence times\textsuperscript{44}. 

![Fig.4.4: Schematic of the Josephson parametric amplifier which will be characterised in this thesis](image)
5. The Pump Tone Compensator System

Pump Tone Cancellation

Before we begin to characterise the amplifier, I needed to address a persistent problem inherent to the design of our apparatus, substantially limiting its performance. As outlined in the previous sections, the operating point of the parametric amplifier at which large signal gains can be achieved, requires the system to be driven by a strong pump signal which can have several negative side-effects on the amplifier:

1. The pump tone is in the order of 60dB stronger than the weak quantum signal that we are attempting to amplify. This can mean that even if we achieve significant amplification of the experimental signals, the measurement sensors of the FPGA may saturate due to the pump tone, and we would be unable to detect the amplified signal clearly.

2. Any small but finite leakage in the circulator or other components used to isolate the signal source from the signal output, will allow part of this very strong pump tone to be injected back into the signal source channel. Due to the strong pump, even small leakage, can cause comparatively large signals to interfere with the source of the quantum signal (fig.5.1).

3. To compensate for this, our cryogenic circuitry includes a secondary line, which we can use to inject a second part of the same pump signal to cancel out the strong pump tone (blue). This requires highly accurate amplitude and phase modulation of the cancellation signal, to match the pump signal which is reflected from the parametric amplifier (red).

To implement this we have used a Marki IQ-0307, IQ mixer which splits the incoming signal into two components 90° out of phase with each other, biases one or both quadrature components and then recombines them. In this way by controlling the DC voltage levels at the bias inputs, we can sweep a field of IQ parameters which represents an area of the signals quadrature plane. The amplitude and phase which will minimize the pump will be a point somewhere on the IQ parameter field and by measuring the power output of the amplifier, we can measure the effectiveness of the compensation.

Fig.5.1: The strong pump tone (red) leaking back into the signal source and the cancellation signal (blue)
This compensation point is visible in the experimental results shown in Figure 5.2. The mixer will give its ideal performance when the signal power entering the mixer is near its preferred operating point, in this case around 15-17dB (top left).

However, the operating point of the paramp may require a weaker pump signal to be used, to produce a specific gain value. It can be observed in the subsequent graphs that as the signal becomes weaker the IQ-response of the mixer becomes skewed away from the ideal, but still exhibits a clear and distinct minima, which gives us the coordinates that we can hone in on when we are optimizing the power output.

There is an important additional complication to this cancellation procedure. Leakage of the strong cancellation tone from the directional coupler (Fig5.3, left arrow) due to the couplers finite isolation rating, can combine with the pump signal to shift the parametric amplifier to a higher-power operating point. This can both interfere with the gain performance of the amplifier, and create a feedback-like situation where the pump tone combines with leakage, which requires a stronger compensator signal, which creates more leakage and so on.

To compensate for this effect, I will be implementing an iterative, parameter-tuning algorithm that seeks out the equilibrium point at which a particular chosen value of gain can be stably maintained, with the maximum amount of pump cancellation available for that gain value. This will be the amplifier operating point at which the feedback cycle has reached a stable equilibrium at which point any further feedback effects are small enough to be considered negligible.

Once operating reliably, this system will allow me to more accurately characterise gain and noise profiles in subsequent stages of the project, and will need to be incorporated into the standard calibration process to regularly reset the operating point of the device, every few measurements, to avoid incremental parameter-creep in the operating point of the amplifier.

Fig.5.2: Map of IQ-tuned cancellation measurements

Fig.5.3: The compensator tone leaking back into the amplifier, the additional power shifting the operating point and gain level.
Simulating the Cancellation Tuning Algorithm

To assist with developing the final cancellation procedure to be used in the laboratory, I have written the Matlab scripts and functions that are included in the Appendix. Basically they are a direct implementation of the derivations presented in the Input-Output theory section. The system simulates the following components so as to directly mirror the complicated dynamics of the real parametric amplifier system.

1. The Simulated Input/Output Response of the Parametric Amplifier

The parametric amplifier is initially modelled as a S11 scattering coefficient that takes the form of equation 3.13 which describes the amplitude and phase response of the reflected output signal as the nonlinear resonator system is driven into higher power regimes, which exhibit linear, non-linear and bilinear behaviours. The simulated input signal combines the weak signal, the strong pump tone and the leakage of the compensator tone, however the dynamics of this stage are mostly controlled by the driving of the pump tone.

As can be seen in the simulated data of Figure 5.4a, the average number of pump photons in the resonator relative to the incident pump power, \( n \), is closely dependent on the detuning between the pump frequency and the resonator frequency, \( \delta \). The figures show how for different values of the drive power and nonlinearity product \( \xi \), the system is driven into increasingly non-linear regimes. As the critical values of \( \delta \) and \( \xi \) are exceeded the model that defines the simulated data becomes unreliable, indicated by the rapid escalation of \( n \).

As the effective drive strength, \( \xi \) is increased it can be seen that the system is driven into increasingly non-linear regimes, reaching a critical values of \( \xi \), beyond which the bilinear region begins. Similarly from this graph we can also clearly see the critical detuning between the resonator and the pump frequency, \( \delta=-\sqrt{3}/2 \), giving us an upper limit of subcritical \( \delta \) and \( \xi \) values, at which the maximum gain can be reached.

Similarly the simulated data in Figure 5.4b shows how much of the incident signal power is absorbed by being reflected from the parametric amplifier, as the loss coefficient \( \gamma \) is increased. For the ideal no-loss case of \( \gamma=0 \), all of the signal is reflected \( |\Gamma|=1 \), while the other extreme, in which \( \gamma=\kappa \), the signal is entirely absorbed by the amplifier at resonance, \( |\Gamma|=0 \).

Figure 5.4: (a) The average number of pump photons in the resonator, relative to the incident pump power, \( n \) as a function of the detuning between pump and resonator frequencies, \( \delta \). (b) The absolute value of the reflection coefficient as the resonators loss coefficient is increased.
2. The Simulated Microwave Circuit Components

The microwave circuit that is coupled to the non-linear resonator has been approximated within the simulation, by incorporating the scattering matrix relations of each of the passive components listed below. As can be seen in Figure 5.5, the scattering matrices allow us to map out the path of the simulated input signals as they are split, interact and combined with the compensator signal. It also allows us to include internal reflections due to inexact impedance matching between experimental components, and circuit attenuation effects by simply modifying a few matrix parameters.

The directional coupler is modelled by the unitary scattering matrix in equation 5.2a. The factor c is the coefficient associated with the ports which are coupled within the device, while the factor d coefficient indicates the connections which should be isolated but may exhibit leakage phenomena. These are related to the characteristic ratings of the directional coupler by:

\[
\begin{align*}
\text{Coupling Value (dB), } C &= -10 \log_{10} |c|^2 \\
\text{Isolation Value (dB), } D &= -10 \log_{10} |d|^2 
\end{align*}
\quad (5.1)
\]

The lengths of cable throughout the system represent a frequency dependent phase delay, calculated as the length of time it takes for the signals to propagate through the dielectric material of which the transmission line is composed. The scattering matrix representation of this is shown in equation 5.2b, where the delay of the cable in seconds is given by: \( \Delta t = \frac{L_{\text{cable}}}{c/n} \). L is the length of the cable, c is the speed of light and n is the refractive index of the dielectric material.

\[
S_{\text{Directional Coupler}} = \begin{bmatrix}
0 & \sqrt{1 - |c|^2 - |d|^2} & jc & -jd \\
\sqrt{1 - |c|^2 - |d|^2} & 0 & -jd & jc \\
jc & -jd & 0 & \sqrt{1 - |c|^2 - |d|^2} \\
-jd & jc & \sqrt{1 - |c|^2 - |d|^2} & 0
\end{bmatrix}
\quad (5.2a)
\]

\[
S_{\text{Cables}} = \begin{bmatrix}
0 & e^{i\Phi \Delta t} \\
e^{i\Phi \Delta t} & 0
\end{bmatrix}
\quad (5.2b)
\]
3. The Simulated Signal Gain Profile of the Parametric Amplifier

The signal gain relation 3.27 detailed in the input-output theory section has been incorporated into a Matlab function which can map the gain profile of the signal in different power regimes. This can be seen in Figure 5.6, which indicates that as the pump power increases the pump frequency detuning $\delta$ at which the maximum gain is achieved shifts linearly, before reaching the critical point of bilinearity after which the model that defines the simulated data becomes unreliable.

The same model also indicates the gain profile of the weak signal as it is detuned around the pump frequency, the different gain traces indicate increasing pump power, which drives the gain up to a peak value before falling back to zero gain. The high power level required by the simulation to reach its gain point are a product of the initial $\gamma$ and $\kappa$ values that I tested the simulations inputs and outputs with, extracted from the experimental work of other groups. However as can be seen from the simulated measurements the system dynamics are largely unchanged by the scalability of the power parameter.

![Signal gain as a function of frequency detuning with increasing pump power up to critical $\delta/\xi$ point.](image)

![Signal gain as a function of the signals detuning from the pump frequency, for a range of pump power values.](image)
4. The Compensator Optimization Algorithm

By setting the fixed system parameters to a range of real-world values, we can use the signal gain function and the signal input/output system to test out the compensator algorithm that we would use in a real laboratory context. Each of these simulated measurement systems has a corresponding goal that we would like to meet, respectively locating and tuning the target gain point and minimizing the output power. The three control parameters which we will use to optimize the simulated system are the compensator signal modulation factors I and Q, as well as the driving power of the pump signal.

We can visualize the accessible coordinates as a set of gain and power output parameter spaces, defined by the I&Q mixer settings and the input power settings. These simulated parameter spaces are shown in figure 5.8, and will allow us to fine-tune the optimization search algorithm. It is important that the optimization works reliably as in an experimental environment we won’t necessarily be able to measure the entire parameter space and will have to trust that the optimization procedure is working correctly.

To find an optimization compromise point between these two goals, we combine them into a single objective function (eqn 5.3), which accesses simulated measurements from both parameter spaces.

\[
f(G, P_{out}) = \log(|G - G_{target}|) - u \log\left(\frac{P_{out}}{P_{signal}}\right)
\]  

(5.3)

The factor u can be fine-tuned depending on the relative importance of the target gain or output minimization condition, finding the correct value of u which suits your specific lab equipment often involves some trial and error, but once it has been located the optimization should work effectively without further tuning, unless the apparatus is substantially reconfigured.

By implementing an iterative, non-linear optimization algorithm based on the Nelder-Mead, downhill-simplex method, we can search the 3 dimensional, I-Q-P parameter space for a coordinate that will efficiently minimise the objective function. Since the final experimental context in which we will be using the procedure has an objective function that is noisy and non-linear, I have use a derivative-free algorithm that could help to minimize measurement load. The speed and efficiency of the calibration is an important criteria for its practical application in the laboratory.
Due to the volatile nature of the unconstrained Nelder-Mead search algorithm, I have followed a three-stage method to pre-process its initial starting point:

1. From the initial guess coordinates, optimize the output power using I-Q coordinates only, staying on the flat plane of constant pump power.
2. Tune the pump power to a point as close to the target gain as possible, with fixed I-Q coordinates.
3. Now that the starting point is in the neighbourhood of the coordinates that we want, perform a full optimization of all three control parameters, to find the global minima within the tolerance of the search algorithm.

This methodology has proved reliable in practice for ensuring that the algorithm locates appropriate minima irrespective of initial starting points. Examples of the results of this optimization process are shown in figure 5.9 for various initial conditions. I have displayed the search data points in the objective function parameter space, once again defined by the I,Q and input power settings. The blue isosurfaces throughout the 3D space represent regions in which the coordinates are close enough to the target gain (rather than the maximum gain) to be within a preset tolerance level and simultaneously close enough to the power output minima to cancel a significant amount of the pump tone.

Realistically any of the blue minima coordinates would be a suitable setting for the experimental apparatus to be optimized, but this means that the more difficult to access objectives, such as very high gain states have fewer available minima and are more difficult to optimize for. As can be seen from the graph, the optimization algorithm reliably finds optimized coordinates that conform to the minima of the objective function, starting points are green circles and ending convergence points are indicated by gold stars..

![Fig 5.9: Examples of simulated optimization search routines run in the objective function, IQP parameter space. Green circles indicate starting coordinates and stars represent final optimized coordinates.](image-url)
Implementation of the Optimization Algorithm

Having refined and demonstrated the optimization methodology in a simulated context, I have implemented the same procedure using a Python-QtLab notebook, which is able to access and control the experimental measurement apparatus that we use at the Superconducting Quantum Device Laboratory.

We have access to two main methods for taking the data that the optimizer needs to function, interfacing with the XtremeDSP Field Programmable Gate Array (FPGA) that is our main measurement device, or to take sweep measurements using a Rohde & Schwarz FSV Spectrum Analyzer. Having adapted the script to both and demonstrating successful convergence on optimized values, I have found that the R&S FSV offers a much faster optimization with more than sufficient accuracy to hone in on a target minima of the objective function. Speed is of great practical importance during calibrations so that they can be easily incorporated into the procedure of measurement routines. However the FPGA based script also performed adequately and may be more convenient in many configurations of the apparatus components.

As can be seen in the example below of an experimental result, I had nominated a moderate level of 25dB gain for the objective function, that should be easy for the parametric amplifier to achieve over a relatively broad band of frequencies in the order of 9 MHz, as can be seen in figure 5.10. I first took a measurement of the power output spectrum with the IQ mixer set to zero bias which can be seen in figure 5.11a and remeasured the same spectrum after the optimization algorithm had found minimized coordinates and reset the amplifiers I, Q and input power settings to this set of coordinates (figure 5.11b).

After repeating this process for different starting points and different initial power levels, I found that the optimization algorithm managed to stabilize the gain level to within ±0.15dB on either side of the 25dB gain target. In addition the power output minimization could routinely achieve a reduction in the order of 20dB of output pump tone. I believe that this reduction could be fine tuned to a higher level and is dependent on the amount of power that is required to reach a particular gain target.

![Figure 5.11: (a) The un-optimized power output spectrum (b) the optimized power output spectrum with -20dB reduction.](image-url)
While it is very likely that even greater output power cancellation is possible with further refinement, I believe that the stabilization of the gain combined with the substantial reduction in output power constitute a proof of principle of the optimization methodology and will prove a useful procedure for future experiments.

Fig. 5.10: Example of optimized gain profile measurement for a target value of 25dB.
6. Linear Response and Tunability

Characterising the Resonator

Throughout this characterization, I have followed the methodology common to several of the groups mentioned in section 4\textsuperscript{25,29}. The first measurement to characterise the parametric amplifier is to map the response of the device’s resonant frequency to changes in the magnetic flux bias applied to the SQUID array. This is achieved using the measurement configuration indicated in figure 6.1, while operating in the linear, low-pump power regime. The pump signal is split into two paths; one path will provide phase corrections for the source, while the other path will interact with the parametric amplifier.

The local oscillator (LO) signal indicated in the diagram is set to a constant offset from the signal frequency, in our case a 25MHz offset. When the signal and LO combine at the IQ Mixer, the signal is down-converted so as to be able to infer both amplitude and phase information from the FPGA measurements. However, as the probe’s initial phase changes each time the source frequency is changed, the system requires the phase reference, line 1 (also down-converted) to make small phase corrections for the difference between the LO and the source phase.

The FPGA measures the complex values of each channel, extracting the reflection coefficient information, by multiplying the signal component $S_1=X_1+iP_1$ with the complex conjugate of the LO component $S^*_2=(X_2+iP_2)^*$. Since the LO reference has a constant amplitude, and the two signals have been corrected such that they are in-phase, the LO tone is cancelled out leaving the reflection coefficient. An additional frequency-dependent distortion effect occurs due to path length differences between the two probe signals. This phase delay can be removed by rotating the measured signal by a factor of $e^{i\omega t}$, which can then be fine-tuned to remove regular oscillations from the measurements.

As can be seen from the example in figure 6.2, the real and imaginary components of the reflection coefficients are then fit to the theoretical model, derived in section 3:

$$\Gamma = \frac{\kappa}{\kappa+\gamma} \frac{1}{2} \left( \frac{\omega_{\text{ref}} + i\omega}{\kappa+\gamma} \right) + 1. \quad (6.1)$$

By varying the external flux applied to the SQUID array, we can shift this resonance point over a range of frequencies, as described in section 1. By measuring and fitting these reflection coefficient responses over a range of applied flux values, we can map the tunability of the resonance frequency of the specific device that we are using in the Superconducting Device Laboratory apparatus.
The range of tuneable frequencies that are accessible by varying the flux can be seen in figure 6.3a, which is a graph of the phase of the signal reflected from the parametric amplifier. As the probe signal reaches the resonance frequency of the resonator it undergoes a strong phase shift over a narrow range of frequencies.

The resonance frequency values extracted from these reflection coefficient measurements, closely follow the theoretical model described by the cavity mode equation described by Wallquist et al.  

\[
k_jd \tan(k_jd) = \frac{ld}{\Phi_0^2} \equiv \frac{ld}{L_j}
\]

Where \(k_j\) is the wave number of mode \(j\), \(d\) is the length of the transmission line resonator, \(l\) is the inductance per unit length of the resonator, \(E_j\) is the Josephson energy of the SQUID, and \(L_j\) is the Josephson inductance of the SQUID.

This equation describes a series of sweeps across the available frequency range, each time the SQUID array receives a magnetic flux quantum of energy equal to \(\phi_0=h/2e\).

By fitting the measured data to this model, I have extracted the maximum Josephson energy of the SQUID array is

\[
E_{J,\text{max}} = 5.009 \times 10^{-22} \text{ J},
\]

or equivalently in the common form:

\[
E_{J,\text{max}}/h = 0.76 \text{ THz}.
\]

From this derived value we can calculate the corresponding Josephson inductance of the array:

\[
L_j = \frac{\Phi_0^2}{E_j} = 8.536 \times 10^{-9} \text{ H},
\]

as well as the critical current of the Josephson junctions, which is equal to:

\[
I_c = \frac{E_j 2\pi}{\phi_0} = 1.522 \mu\text{A}.
\]
Q Factor Measurements

In addition to the information that can be extracted from the resonance frequency, the values of $\kappa$ and $\gamma$ that were fitted to the individual reflection coefficient measurements (such as figure 6.2), can be used to characterize the resonators quality factors (Q). The internal Q factor is a result of losses within the resonator itself, and can be described by the equation $Q_{\text{int}} = \frac{\omega_o}{\gamma}$ and is directly proportional to level of gain that can be achieved at that setting. The external Q factor, $Q_{\text{ext}} = \frac{\omega_o}{\kappa}$, is primarily a result of the coupling coefficient of the parametric amplifiers coupling capacitor, but is also affected by the characteristic impedance of the external transmission line, and the configuration of components in the circuitry.

As can be seen from the quality factor data in figure 6.4a, we can see a definite trend in the order of a quality factor of 1000, in the main operating flux point region, although as indicated by the errorbars, the $\kappa$ value was significantly more difficult to accurately determine. This internal Q factor is designed into the structure of the parametric resonator, but is often strongly affected by inaccuracies in the etching and fabrication process.

The external Q factor (Fig6.4b) has a relatively stable, low Q factor value, which can be a desirable design feature as it indicates a broad linewidth of the resonance peak. As indicated by the overall trend of the graph, at the highest frequency limit of the SQUIDs flux-tuning (around -7.4V), the coupling capacitor becomes comparatively more transparent to the reflected signal and the external Q factor drops to a minima.

The combined Q factor of the system in 6.4c, \[ \frac{1}{Q_{\text{total}}} = \frac{1}{Q_{\text{int}}} + \frac{1}{Q_{\text{ext}}} \], is dominated by the low external Q factor, indicating a broad overall bandwidth.

Fig.6.4: The quality factors derived from fitting reflection coefficient data, for (a) Internal Q, (b) External Q and (c) Total Q of the resonator.
Measuring the Optimal Pump Power Operating Point

As described in section 2 the parametric amplifier can operate in several different modes, from the linear, to the non-linear and the high-power, bilinear regime. I have mapped out the effect of increasing the drive power on the phase of the reflection coefficients and the results can be seen in figure 6.5a.

The system exhibits relatively gradual, linear behaviour at the lower powers. But as the driving level of the pump is increased the reflection coefficient displays an increasingly steep drop in phase at the relevant resonant frequency for that power setting. This is easiest to see in the individual traces compared in figure 6.5b.

These traces are representative of slices taken through the power mapping measurements of figure 6.5a, and constitute a direct measurement of the nonlinearity described by the Duffing oscillator model discussed in section 1. The measurement is such that we cannot directly measure the multiple solution curve which defines the bilinear regime, but we can infer this region from the instantaneous drop exhibited by some of the measured curves (around 1dBm in figure 6.5b).

The graph in figure 6.5c is the transfer function of the amplifier (similar to the theoretical model in figure 1.4), which as described in section 1, shows how, when operating in a highly nonlinear region of the transfer function curve, even small perturbations in the input signal will cause a disproportionately large phase response in the reflected signal which is measured.

Fig.6.5: (a) Nonlinear phase response of reflected pump with increasing driving power. (b) Individual normalized phase traces, taken at different power levels of the power-frequency graph. (c) Amplifier transfer function showing the normalized reflected phase as a function of input power, for a range of frequency detunings around the critical operating point frequency.
Calculating the Kerr Nonlinearity

Following the method described in section three (equation 3.18), I have used the parameters measured above and the fact that there are ten SQUIDs fabricated onto this device, to calculate a value for the constant of \( K = -4.96 \times 10^5 \pm 1.22 \times 10^5 \), often expressed in the form: \( \frac{K}{\omega_o} = -6.81 \times 10^{-6} \pm 1.68 \times 10^{-6} \).

This value allows the calculation of the theoretical power level at which a system operating with similar parameters should reach its critical point. Due to reconfigurations of our apparatus, the critical input power level that was observed during my nonlinearity and gain measurements, were in the order of 6.3dBm. Once the combined attenuation effects of the directional coupler, splitters, and attenuators were quantified to be in the order of a -70dB reduction. The effective driving power delivered to the amplifier was determined to be approximately, \( P_{\text{critical}} \approx -64\text{dBm} \).

This can be compared to the theoretical power level at which the system should reach critical nonlinearity which is given by the equation: \( P_{\text{critical}} = \hbar \omega_{\text{pump}} (\kappa + \gamma)^3 / \sqrt{27} \kappa |K| = 0.83\mu \text{V} = -61.64\text{dBm} \). Allowing for the possibility of overlooked or over-estimated aspects of the microwave network attenuation, this is an acceptably close comparison between the theoretical model and the amplifiers true performance.
7. Measurement of the Parametric Amplifier Gain

Maximum Gain Measurements

The resonator measurements performed in the previous section give us a clearer idea regarding the operating power and frequency range over which parametric amplification will occur, and allow us to locate the region of highest possible gain. The next part of the characterisation involves the measurement of the amplifiers gain and the corresponding bandwidths over which it can be achieved. To do this we first map the maximum possible signal gain over a range of signal frequencies and pump energies. We combine the strong pump tone with a weak probe signal that is set to a power in the order of 50-60dB weaker than the pump source and shares a common phase reference signal with the pump and LO-detuning sources.

The maximum gain possible is achieved when the signal frequency is set as close as possible to the pump frequency, without the signal being overpowered by the bandwidth of the pump tone. The method that we use to measure the maximum gain across a wide range of possible signal frequencies is to set the pump to a constant 10kHz offset from the swept signal frequency, and as with previous measurements to set the down-conversion LO-tone to a 25 MHz offset from the signal.

Since this 10kHz offset is much smaller than the peaks bandwidth, it is sufficiently close to the high-gain peak of the gain profile to be assumed to be set to the same frequency. This specific offset also performs an important function, filtering the pump tone out of the FPGA measurements through an averaging process. For our apparatus the trigger time for each measurement sequence is set to a constant 25μs window which corresponds to a 40kHz triggering frequency. By choosing each pump frequency that is measured to be a multiple of this 40kHz frequency, every measurement period changes the pump signals phase by π/2, which when averaged over time causes them to cancel out. The signal is set to a 10kHz offset, which is unable to be cancelled out by the same filtering process, leaving it unaffected.

The measurements of the maximum gain achievable at different pump power levels, for a fixed magnetic flux bias value, is shown in figure 7.1. This graph is of great practical benefit in an experimental context, showing a window of system parameters at which a chosen gain level can be accessed. The corresponding gain profiles over a range of pump power levels are displayed in figure 7.2.

![Graph showing measurement of peak signal gain with increasing pump power.](image)
Each of these maximas represent the narrowband, maximum gain available at that pump power level, and as the theoretical model suggests, the increasing effective drive power $\xi$, causes the critical point of maximum gain to be shifted by a corresponding change in the pump frequency detuning $\delta$. As the pump power is increased beyond the 6.45dBm maximum peak shown above, the peak detunings continue to be at lower frequencies, but the signal gain achieved reduces back to the background level of the system.

**Measurement of the Gain Bandwidth**

Once we have located this small window of parameters that will achieve the maximum gain, we can measure the bandwidth over which amplification can be achieved. This is an important factor that will determine how we use the parametric amplifier in practice, as for some experiments we will need to detune the device to a broad bandwidth and for others we will need the maximum gain over a much more narrow range of frequencies.

![Image of Gain Bandwidth Profiles](image-url)

**Fig.7.3: Gain bandwidth profiles at several different pump power levels**
For this measurement, the pump power is set to the preferred gain level indicated by our measurements in figure 7.1, and the pump frequency is fixed to the resonant frequency around which the maximum gain occurs. We then sweep the weak signal around this fixed pump frequency and measure the resulting gain as a function of Δ, the signals detuning from the fixed pump frequency.

To maintain the pump filtering effect the signal and pump are still detuned by 10kHz from each other and the pump frequencies and the frequencies measured are multiples of the 40kHz triggering frequency. These experimental gain-bandwidth measurements follow the theoretical models Lorentzian form (see figure5.7) very closely. The measurements are shown in figure 7.3 and detailed in the table in figure 7.4a.

As can be seen in figure 7.4b, there is a general trend in the relation between bandwidth and gain that follows the expected inverse relation of $\sqrt{G_0} B = \alpha (\kappa + \gamma)$ where B is the bandwidth, (κ+γ) is the linewidth and $\alpha$ is a constant scaling factor.

These measurements of the wide range of available maximum gain values and the reliability of the parametric amplifier to find, stabilise and maintain strong gain over a tuneable bandwidth suggests that it will prove to be an important tool in upcoming qubit experiments.
Conclusion

In this thesis I have presented the broad theoretical background of the Josephson parametric amplifier, as well as demonstrating a detailed spectroscopic analysis of the device, characterizing the parameters and operating regimes that are required to operate it optimally. Measurements of the amplifiers response to input probe signals have been in close agreement with the theoretical model I have described and the levels of signal gain and amplification bandwidth that I have measured have been favourably comparable to any of the contemporary groups that I have outlined in the literature review section.

In addition, I have simulated and implemented a solution to the pervasive problem of excessive pump power in the output signal, which is a particular problem in Josephson parametric amplifiers that share similarities with our relatively common design configuration. I believe that the calibration routine that I have used would be of practical use in further experiments, to reduce the instability of the amplifiers operating point due to internal power leakage sources.

Josephson parametric amplifiers offer a range of exciting applications such as exceeding the Standard Quantum Limit of amplification noise, generating entangled signal and idler photons and exhibiting squeezed vacuum fields. But as I have demonstrated in this honours project it is also a powerful practical tool for the efficient amplification of weak quantum fields for classical detection schemas.

With the ongoing development of superconducting ‘on-chip’ experimental devices, as well as the developing field of microwave-optical hybrid apparatus, it seems that the necessity for parametric amplification technology will only increase in coming years.


Appendix: Matlab Code for Parametric Amplifier Simulation

There are two scripts, one which visualizes the data space and the other maps the minimization routine:
IQP_Mapping.m, IQP_Minimization.m

These call upon three functions, which perform the minimizations:
IQ_Outputminimize.m, P_Gainminimize.m, IQP_minimize.m

These call upon two functions which describe the gain and power dynamic of the parametric amplifier:
PowerOutFunction.m, GainFunction.m

IQP_Mapping.m

%PARAMETER SPACE MAPPING SCRIPT - "IQP_Mapping.m"
%This script visualizes the 3D parameter space of (I,Q,P) coordinates
%using several different graphical views, such as isosurfaces and
%transparent plane layers through the bounded space.

%If this script is run first and 'hold on' is left on, running the
"IQP_Minimization.m" script will add the search iteration points onto
%the graph of the visualized space

clc;
clear all;

Gtarget=200; %This is the Target Gain that will define the obj.function
delta=0; %the signal-pump detuning (0=max gain)
d=-0.82; %the nominated pump-resonator detuning delta
k=1.45744e7; %coupling capacitor coefficient
g=2.506e6; %internal loss coefficient
K=-6893.7; %Kerr non-linear coefficient
B6=0; %The input from the output port if this happens

Coupling=20; %db Directional Coupler -> the negative is in the conversion
Leakage=Coupling+20; %20dB weaker than the coupling rate
teflon=1/sqrt(2.5); %cable material coefficient, fraction of light speed
cablelength0 =0.0; %metres, cable between directional coupler & paramp
cablelength1 =0.0; %metres, cable after the output of the directional coupler
teflonmat=telfon; %transmission line cable material

Freqs=[6.249e9 6.269e9 1e5]; %LOW FREQ - HIGH FREQ - SIZE OF STEPS
frequence=Freqs(:,1):Freqs(:,3):Freqs(:,2); %List of frequencies
resonatorfreq=6.2670e9; %Frequency of the transmission line resonator
LRange=[132 134 8]; %LOW POWER - HIGH POWER - NO. OF STEPS %132.2-132.445
P=LRange(1):((LRange(2)-LRange(1))/LRange(3)):LRange(2);
Q=-1:0.0.04:1; %Compensator mixer tuning parameters
I=-1:0.04:1;

Output_List=[];
Gain_List=[];
Rating_List=[];
ShortAList=[];
ShortBList=[];

    for p=1:length(P)
        for i=1:length(I)
for q=1:length(Q)
    [PowerOut, STDevPowerOut, SignalGain, Rating] = ...
    IQPObjFunction( Gtarget, delta, I(i), Q(q), P(p), ... 
    d, k, g, K, resonatorfreq, freqrange, Coupling, Leakage, ... 
    B6, cablelength0, cablelength1, vmaterial );
    Output_List=[Output_List;PowerOut];
    Gain_List=[Gain_List; SignalGain];
    Rating_List=[Rating_List; Rating];
end

disp(length(P)-p)
end

Output_Data=reshape(Output_List,length(Q),length(I),length(P));
Gain_Data=reshape(Gain_List,length(Q),length(I),length(P));
Rating_Data=reshape(Rating_List,length(Q),length(I),length(P));

%MULTILAYER GRAPH
figure;
hold on
for pwr=1:2:length(P) %Enter how many planes you want here
    surf(I, Q, ones(length(I), length(Q)) * P(pwr), Rating_Data(:, :, pwr), 
    'EdgeColor', 'none');
    alpha(0.3); %Control the transparency of the planes
end
xlabel('I Offset (V)');
ylabel('Q Offset (V)');
zlabel('Pump Power (dBm)');
view(3);

%ISOSURFACE GRAPH
figure;
hold on;
[x, y, z] = meshgrid(I, Q, P);
%By changing the fifth argument number below, you can select the value
isosurface(x, y, z, Rating_Data, 19); %of the isosurface that is mapped
xlabel('I Offset (V)');
ylabel('Q Offset (V)');
zlabel('Pump Power (dBm)');
% This script calls the three system dynamics functions and collects and graphs the results. If IQP_Mapping has already been run, and 'hold on' is on, the points will add to that graph. Otherwise uncomment "figure" and "hold on" lines.

Gtarget=250; % The target gain which will define the rating coordinate points

% These are the initial guesses for power and I/Q coordinates
PowerInit=132.5;
IQcoords=[0.6,0.378];

% Step 1: Perform a 2D IQ amplitude-output minimize at the nominated power level guess
[IQdata] = IQ_Outputminimize(PowerInit,IQcoords);

% Step 2: Perform a 1D P minimize towards the target gain, loading the last IQ coords
[Pdata] = P_Gainminimize(Gtarget,IQdata.coords(length(IQdata.coords),:),PowerInit);

% Step 3: Loading the last IQcoords and Pcoords, perform the final IQP minimization around this point
[Optdata] = IQP_minimize(Gtarget,IQdata.coords(length(IQdata.coords),:),Pdata.coords(length(Pdata.coords),:));

IQpts=[IQdata.coords ones(length(IQdata.coords),1)*132.5];
Ppts=[ones(length(Pdata.coords),1)*(IQdata.coords(length(IQdata.coords),1)) ...
     ones(length(Pdata.coords),1)*(IQdata.coords(length(IQdata.coords),2)) ...
     Pdata.coords];
Opts=Optdata.coords;

% plotting step 1 search iterations
plot3(IQpts(1,1),IQpts(1,2),IQpts(1,3),'g^','MarkerSize',20)
plot3(IQpts(:,1),IQpts(:,2),IQpts(:,3),'g-')
plot3(IQpts(:,1),IQpts(:,2),IQpts(:,3),'g+')

% plotting step 2 search iterations
plot3(Ppts(:,1),Ppts(:,2),Ppts(:,3),'r-')
plot3(Ppts(:,1),Ppts(:,2),Ppts(:,3),'r+')

% Plotting step 3 search iterations
plot3(Opts(:,1),Opts(:,2),Opts(:,3),'b-')
plot3(Opts(:,1),Opts(:,2),Opts(:,3),'b+')
plot3(Opts(length(Opts),1),Opts(length(Opts),2),Opts(length(Opts),3),'b*','MarkerSize',25)
view([90 0 0]);
caxis([  ]);
function [Optdata] = IQP_minimize(Gtarget, IQcoords, Pcoord)

% This function performs the final IQP minimization once the IQ_Outputmin
% and P_Gainmin functions have been run to set the initial coordinates.

% DEFINITIONS
delta = 0; % signal detuning from pump (0=max gain)
d = -0.82; % pump detuning from resonator frequency
k = 1.45744e7; % coupling capacitor coefficient
g = 2.506e6; % internal loss coefficient
K = -6893.7; % Kerr nonlinearity coefficient
B6 = 0; % The input signal at the output port, in case that happens

Coupling = 20; % dB Directional Coupler -> the negative is in the conversion
Leakage = Coupling + 20; % 20 dB weaker than the coupling rate

teflon = 1/sqrt(2.5); % material of cable, fraction of light speed

Freqs = [6.249e9 6.2609e9 1e5]; % LOW FREQ - HIGH FREQ - SIZE OF STEPS
freqrange = Freqs(:,1):Freqs(:,3):Freqs(:,2); % Frequency list
resonatorfreq = 6.2670e9; % Fixed frequency of resonator

Optdata.coords = []; Optdata.fval = []; Optdata.i = [];

% MINIMIZE THE OBJECTIVE FUNCTION USING A NELDER-MEAD ALGORITHM
options = optimset('OutputFcn', @outfun); Ocoords = [IQcoords Pcoord]; [Ocoords, fval, exitflag, output] = fminsearch(@IQPObjFuncReduced, Ocoords, options)

% THE OBJECTIVE FUNCTION

function [ Rating ] = IQPObjFuncReduced( Ocoords )

I = Ocoords(1); Q = Ocoords(2); P = Ocoords(3);

[ SignalGain, GainRating ] = GainFunction( Gtarget, delta, I, Q, P, d, k, g, K, resonatorfreq, freqrange, Coupling, Leakage, B6, cablelength0, cablelength1, vffieldmaterial1, vffieldmaterial2, vffieldmaterial3, vffieldmaterial4 );

[ PowerOut, STDevPowerOut ] = PowerOutFunction( I, Q, P, d, k, g, K, resonatorfreq, freqrange, Coupling, Leakage, B6, cablelength0, cablelength1, vffieldmaterial1, vffieldmaterial2, vffieldmaterial3, vffieldmaterial4 );
alpha_s = sqrt(50.*(10.^((P-60)/10))./1000);

% Choices of Objective Functions, can add new ones here to find different rating systems
% Rating = (max(10*log10(STDevPowerOut), 10+10*log10(PowerOut)) + 2*max(-17, 10*log10(abs((SignalGain/Gtarget)-1))));
Rating = log10(GainRating) + log10(PowerOut/(SignalGain*alpha_s));

end
function stop = outfun(Ocoords,optimValues,state)
stop = false;

switch state
    case 'init'
    case 'iter'
        % Save the data into variables, coords must be a row vector.
        Optdata.fval = [Optdata.fval; optimValues.fval];
        Optdata.coords = [Optdata.coords; Ocoords];
        Optdata.i = [Optdata.i; optimValues.iteration];
    case 'done'
    otherwise
end
end
end
function [IQdata] = IQ_Outputminimize(PowerInit, IQcoords);

%DEFINITIONS
IQdata.coords = [];
IQdata.fval = [];
IQdata.i=[];

%MINIMIZE THE OBJECTIVE FUNCTION USING A NELDER-MEAD ALGORITHM
options=optimset('OutputFcns',@IQoutfun,'TolFun',1e-15,'TolX',1e-15);
[IQcoords,fval,exitflag,output]=fminsearch(@IQ_minimize,IQcoords,options);

%THE OBJECTIVE FUNCTION

function [ PowerOut ] = IQ_minimize( IQcoords);
I=IQcoords(1);
Q=IQcoords(2);
P=PowerInit(1);

[PowerOut] =PowerOutFunction(I,Q,P);
end

%RECORDS THE DATA FROM EVERY ITERATION INTO A COMBINED STRUCTURE

function stop = IQoutfun(IQcoords,optimValues,state);
stop = false;

switch state
    case 'init'
    case 'iter'
        % Save the data into variables, coords must be a row vector.
        IQdata.fval = [IQdata.fval; optimValues.fval];
        IQdata.coords = [IQdata.coords; IQcoords];
        IQdata.i = [IQdata.i; optimValues.iteration];
    case 'done'

    otherwise

end
end
end
%DEFINITIONS

Pdata.coords = [];  
Pdata.fval = [];  
Pdata.i=[];

delta=0;%offset between signal and pump frequency  
d=-0.82;%detuning between pump and resonator frequency  
k=1.45744e7; %coupling capacitor coefficient  
g=2.506e6; %internal loss coefficient  
K=-6893.7;%Kerr nonlinearity coefficient  
B6=0;%Input signal at the output port, if this happens

Coupling=20; %db Directional Coupler -> the negative is in the conversion  
Leakage=Coupling+20; %20dB weaker than the coupling rate  
teflon=1/sqrt(2.5); %material of cables fraction of light speed  
cablelength0 =0.0; %metres, cable between directional coupler and paramp  
cablelength1 =0.0; %metres, cable after directional coupler output  
vmaterial=teflon;%dielectric material in transmission cables

Freqs=[6.249e9 6.2609e9 1e5]; %LOW FREQ - HIGH FREQ - SIZE OF STEPS  
freqrange=Freqs(:,1):Freqs(:,3):Freqs(:,2); %Frequency list  
resonatorfreq=6.2670e9;%Frequency of resonator

%MINIMIZE THE OBJECTIVE FUNCTION USING A NELDER-MEAD ALGORITHM

%options=optimset('Display','iter','OutputFcn',@Poutfun);  
options=optimset('OutputFcn',@Poutfun);  
IQcoords=[-0.1,0.1]; %test values  
Pcoords=132.8;  
[Pcoords,fval,exitflag,output]=fminsearch(@P_minimize,PowerInit,options)

%THE OBJECTIVE FUNCTION

function [ PRating ] = P_minimize( Pcoords )

I=IQInit(1);  
Q=IQInit(2);  
P=Pcoords;

[ SignalGain, PRating ] = GainFunction(  
Gtarget,delta,I,Q,P,d,k,g,K,resonatorfreq,freqrange,Coupling,Leakage,B6,cablelength0,cablelength1,vmaterial );
end

%RECORDS THE DATA FROM EVERY ITERATION INTO A COMBINED STRUCTURE

function stop = Poutfun(Pcoords,optimValues,state)
stop = false;
switch state
 case 'init'
case 'iter'
    % Save the data into variables, coords must be a row vector.
    Pdata.fval = [Pdata.fval; optimValues.fval];
    Pdata.coords = [Pdata.coords; Pcoords];
    Pdata.i = [Pdata.i; optimValues.iteration];
end

otherwise
    end
end
end
function [ PowerOut, STDevPowerOut ] = PowerOutFunction( I,Q,P,d,k,g,K,resonatorfreq,freqrange, Coupling, Leakage, B6, cablelength0, cablelength1, vfmaterial )

% ------------- INPUTS -------------------------------
% I/Q=the quadrature components of the IQ mixer
% P=pump power
% d=pump frequency detuning from resonator frequency
% B6 is the voltage injected into the output of the paramp (usually zero)
% k=kappa, the paramp coupling coefficient to the external circuitry
% g=gamma, the paramp loss coefficient
% K=the Kerr nonlinearity
% resonatorfreq=the resonance frequency of the resonator cavity
% freqrange= frequency range being tested, mostly just to discern pumpfreq
% Coupling=directional coupler coupling in decibels (usually 20dB)
% Leakage=directional coupler leakage in decibels (usually coupling-20dB)
% cablelength0=length of cable between directional coupler and paramp
% cablelength1=length of cable between directional coupler and output
% vfmaterial=the transmission coefficient of the cable material

% PUMP, SIGNAL AND VACUUM POWER TO VOLTAGE CONVERSIONS
alpha_in=sqrt(50.*(10.^((P./10))./1000));  % pump voltage
alpha_s=sqrt(50.*(10.^((P-60)./10))./1000); % signal voltage
alpha_v=sqrt(50.*(10.^((P-100)./10))./1000); % vacuum voltage (max)

% --------------- INPUT SIGNALS ------------------------
% BROAD SPECTRUM SIGNAL
Asignal=ones(length(freqrange),1).*alpha_s;

% BROAD SPECTRUM VACUUM WITH NOISE
Avacuum=rand(length(freqrange),1).*alpha_v;

% SINGLE LINE PUMP WITH BROAD SIGNAL WITH NOISE
pumpfrequency=d*(k+g)+resonatorfreq;
tmp=abs(freqrange-pumpfrequency);
[b idx]=min(tmp); % this selects the closest actual frequency to the freq of the nominated delta value
closest=freqrange(idx);
Apump=(freqrange(:)==closest).*alpha_in;
Aline=Apump+ Asignal + Avacuum;

% COMPENSATOR INPUT SIGNAL
mod_amp=sqrt(Q.^2+I.^2);
mod_phi=atan(Q./I);
Acomp=Apump.*mod_amp.*exp(1i.*mod_phi)+Avacuum;

% --------------- MICROWAVE NETWORK CIRCUIT ------------------
% --------------- BEFORE PARAMP -----------------------------

% Directional Coupler Coupling/Leakage Coefficients
c=1i*abs(sqrt(10.^((Coupling./(-10)))));
l=1i*abs(sqrt(10.^((Leakage./(-10)))));

% Paramp Cable Propagation Scattering Elements
delay0=cablelength0.*(2.9979e8*vfmaterial);
S0_11=0;
S0_12=exp(1i*freqrange'*delay0);
S0_21=exp(-1i*freqrange'*delay0);
S0_22=0;

%Output Cable Propagation Scattering Elements
delay=cablelength1.*(2.9979e8*vfmaterial);
S1_11=0;
S1_12=exp(1i*freqrange'*delay1);
S1_21=exp(-1i*freqrange'*delay1);
S1_22=0;

%Directional Coupler Inputs
A2=B6*S1_21;
A3=Aline;
A4=Acomp;

%Input from Circuit to Paramp Loop Section
B1=sqrt(1-(abs(c)).^2-(abs(l)).^2).*A2+1i*c.*A3-1i*1.*A4;

%--------------------------------------EPSILON ELEMENTS--------------------------------------

%The signal that reaches the paramp and sets the delta and epsilon values
A5=S0_12.*B1;
A5_tilda=(sqrt(k).*A5)./(k+g);

%SINGLE LINE PUMP WITH BROAD SIGNAL EPSILON
e=(((A5_tilda).^2)*K)/(k+g);

%--------------------------------------GAMMA CALCULATION--------------------------------------

%This is the average number of photons in the resonator wrt input pump
%function was generated using Mathematica solver and extensively tested
n=(2.*d)./(3.*e) - ((d.^2 + 1./4)./(3.*e.^2) -
(4.*d.^2)./(9.*e.^2))/(1./3) + ((1./2.*e.^2 + 8.*d.^3)/(27.*e.^3) -
(d.*(d.^2 + 1./4))./(3.*e.^3)).^2 + ((d.*(d.^2 + 1./4))./(3.*e.^3)).^2 + ((d.*(d.^2 + 1./4))./(3.*e.^3)).^2 -
(4.*d.^2)./(9.*e.^2)) + 1./(2.*e.^2) + (8.*d.^3)/(27.*e.^3) -
(d.*(d.^2 + 1./4))./(3.*e.^3)).^2 + ((1./2.*e.^2 + 8.*d.^3)/(27.*e.^3) -
(d.*(d.^2 + 1./4))./(3.*e.^3)).^2 + ((d.*(d.^2 + 1./4))./(3.*e.^3)).^2;

%Reflection coefficient of the Paramp
Gamma=(((k./(k+g)).*(1./(0.5-1i*d+l+1i*e.*n)))-1).*exp(-1i*pi);

%--------------------------------------MICROWAVE NETWORK CIRCUIT--------------------------------------

%--------------------------------------AFTER PARAMP--------------------------------------

%This little loop is to do with the S22 reflection coefficient right next to
%paramp, see map (not significant)
shortS0_22loop=0;
i=0; %number of iterations of the little short-circuit loop next to the paramp
%syms ii; %
shortS0_22loop=S0_12.*B1.*Gamma.*S0_22.*symsum('S0_22.^(ii-1).*Gamma^ii',0,i).*S0_12
A1=S0_11.*B1+S0_12.*B1.*Gamma.*S0_12+shortS0_22loop;
B2=sqrt(1-(abs(c)).^2-(abs(l)).^2).*A1-1i*1.*A3+1i.*c.*A4;
A6=B6.*S1_22+B2.*S1_12;
PowerOut=max(abs(A6));
STDevPowerOut=std(abs(A6));
end
function [ SignalGain, GainRating ] = GainFunction( Gtarget, delta, I, Q, P, d, k, g, K, resonatorfreq, freqrange, Coupling, Leakage, B6, cablelength0, cablelength1, vmaterial)
%This is the optimization tool for indicating how close to the target the
%gain is with the present settings for I, Q, P
%There are a lot of inputs but most of these stay constant:
%-------------------INPUTS-------------------
%Gtarget=target gain
%delta=signal detuning from the pump frequency
%I/Q=the quadrature components of the IQ mixer
%P=pump power
%d=pump frequency detuning from resonator frequency
%B6 is the voltage injected into the output of the paramp (usually zero)
%k=kappa, the paramp coupling coefficient to the external circuitry
%g=gamma, the paramp loss coefficient
%K=the Kerr nonlinearity
%resonatorfreq=the resonance frequency of the resonator cavity
%freqrange= frequency range being tested, mostly just to discern pumpfreq
%Coupling=directional coupler coupling in decibels (usually 20dB)
%Leakage=directional coupler leakage in decibels (usually coupling-20dB)
pumpfrequency=d*(k+g)+resonatorfreq;
tmp=abs(freqrange-pumpfrequency);
[b idx]=min(tmp); %this selects the closest actual "sim-measured" frequency to
%the freq of the nominated detuning value
pumpfreqclosest=freqrange(idx);
alpha_in=sqrt(50.*(10.^(P./10))./1000); %pump voltage
alpha_s=sqrt(50.*(10.^(P-60)./10))./1000; %signal voltage
alpha_v=sqrt(50.*(10.^(P-100)./10))./1000; %vacuum voltage (max)

couple=1i*abs(sqrt(10^(Coupling/(-10))));
leakage=1i*abs(sqrt(10^(Leakage/(-10))));
delay0=cablelength0*(2.9979e8*vmaterial);
delay1=cablelength1*(2.9979e8*vmaterial);

%Compensator signal
mod_amp=sqrt(Q.^2+I.^2);
mod_phi=atan(Q./I);
Acomp1=alpha_in.*mod_amp.*exp(1i*mod_phi)+alpha_v;

%Passing through the microwave circuit network
gainA2=B6*exp(1i*pumpfreqclosest*delay1);
gainA3=alpha_in+alpha_s+alpha_v;
gainA4=Acomp1;
gaininput=exp(1i*pumpfreqclosest*delay0).*(sqrt(1-(abs(couple))^2-(abs(leakage))^2).*gainA2+1i*couple*gainA3-1i*leakage.*gainA4);
gaininput_tilda=(gaininput.*sqrt(k))./(k+g);
e_gain=((gaininput_tilda).^2).*K)./(k+g); %The gain function uses a single e
%value corresponding to the power level, rather than the freqrange version.

GainFunction.m
\[
\frac{(4d^2)(9e_{\text{gain}}^2)^3}{(2e_{\text{gain}}^2 + 1)^3} + \frac{1}{2e_{\text{gain}}^2} + \frac{8d^3}{27e_{\text{gain}}^3} - \frac{d(d^2 + 1/4)}{3e_{\text{gain}}^3}^{1/3};
\]

\[
\lambda_{\text{neg}} = \frac{1}{2} - \sqrt{(e_{\text{gain}}n_{\text{gain}})^2 - (d - 2e_{\text{gain}}n_{\text{gain}})^2};
\]

\[
\lambda_{\text{plus}} = \frac{1}{2} + \sqrt{(e_{\text{gain}}n_{\text{gain}})^2 - (d - 2e_{\text{gain}}n_{\text{gain}})^2};
\]

% Implementation of the signal gain component as described in the theory section
Gain_s = 1 + \frac{k}{k+g} \ast \frac{(1i(d - 2e_{\text{gain}}n_{\text{gain}} - \delta) + 1/2)}{(1i \ast \delta - \lambda_{\text{neg}}) \ast (1i \ast \delta - \lambda_{\text{plus}})};

\]

SignalGain = abs(Gain_s)^2;
GainRating = abs(SignalGain - Gtarget);

end